Summary of the *Standards for Mathematical Practice* and some Associated Prompts & Questions to Develop Mathematical Thinking

Excerpted / Adapted from:
Common Core State Standards, Standards for Mathematical Practices, p. 6-8;
http://katm.org/wp/wp-content/uploads/flipbooks/High-School-CCSS-Flip-Book-USD-259-2012.pdf

The Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x2 + x + 1)$, and $(x – 1)(x3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content
The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction. The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics,

explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices. In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Student Behaviors, and Questions and Prompts Related to the Standards of Mathematical Practices

1. Make sense of problems and persevere in solving them.

Students:

- can interpret and make meaning of the problem looking for starting points.
- can analyze what is given to explain to themselves the meaning of the problem.
- can plan a solution pathway instead of jumping to a solution.
- can monitor their progress and change the approach if necessary.
- can 'see' relationships between various representations.
- can relate current situations to concepts or skills previously learned and can connect mathematical ideas to one another.
- understand various approaches to solutions.
- continually ask themselves, "Does this make sense?"

MP1 - Questions & Prompts

- How would you describe the problem in your own words?
- How would you describe what you are trying to find?
- What do you notice about ...?
- What information is given in the problem?
- Describe the relationship between the quantities.
- Describe what you have already tried.
- What might you change?
- Talk me through the steps you've used to this point.
- What steps in the process are you most confident about?
- What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin?
- How else might you organize, represent, or show ...?

2. Reason abstractly and quantitatively.

Students:

- can make sense of quantities and their relationships.
- are able to represent a situation symbolically and manipulate the symbols (*decontextualize*), and make meaning of the symbols in a problem (to contextualize quantitative relationships).
- understand the meaning of quantities and are flexible in the use of operations and their properties.
- can create a logical representation of the problem.
- attend to the meaning of quantities, not just how to compute them.

MP2 - Questions & Prompts

- What do the numbers used in the problem represent?
- What is the relationship of the quantities?
- What is the relationship between ___ and ___?
- What does ___ mean to you? (e.g. symbol, quantity, diagram)
- What properties might we use to find a solution?
- How did you decide in this task that you needed to use …?
- Could another operation or property be used to solve this? Why or why not?

3. Construct viable arguments and critique the reasoning of others.

Students:

- can analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- can justify conclusions with mathematical ideas.
- can listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- can ask clarifying questions or suggest ideas to improve/revise the argument.
- can compare two arguments and determine correct or flawed logic.

MP3 - Questions & Prompts

- What mathematical evidence would support your solution?
- How can we be sure that …? – or– How could you prove that …?
- Will it still work if …?
- What were you considering when …?
- How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not? How do you know?
- What is the same and what is different about …?
- How could you demonstrate a counter-example?

4. Model with mathematics.

Students:

- ask themselves, "How can I represent this mathematically?"
- can understand that mathematics is a way to reason quantitatively and abstractly, by generalizing beyond a specific situation, and by applying a generalization to a specific context.
- can apply the math they know to solve problems in everyday life.
- are able to simplify a complex problem, and identify important quantities or look at relationships.
- can represent mathematical connections to describe a situation either with an equation or a diagram, and interpret the results of a mathematical situation.
- can reflect on how well a model fits a situation, and can possibly improve on or revise the model.

MP4 - Questions & Prompts

- Would it help to create a diagram, graph, or table?
- What mathematical model could you construct to represent the problem?
- What are some ways to represent the quantities? Is there an equation or expression that matches the diagram, number line, chart, or table?
- Where in the task did you see any of the quantities from your equation or expression?
- What are some ways to visually represent the connections in the situation? What formula might apply in this situation?

5. Use appropriate tools strategically.

Students:

- can use available tools, and recognize the strengths and limitations of each.
- can use estimation and other mathematical knowledge to detect possible errors.
- can identify relevant external mathematical resources to pose and solve problems.
- can use technological tools to deepen their understanding of mathematics.

MP5 - Questions & Prompts

- What mathematical tools could we use to visualize and represent the situation?
- What information are you given? What do you know that is not stated directly in the problem?
- What approach are you considering trying first? Why try *that* first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use: a diagram, a calculator, a graph, number line, ruler, a different manipulative, etc,?
- How was it helpful to use [specific tool]? What can using [tool A] show us that [tool B] may not?
- In what situations might it be more informative or helpful to use [tool A]?

6. Attend to precision.

Students:

- can communicate precisely with others, and try to use clear mathematical language when discussing their reasoning.
- can understand meanings of symbols used in mathematics, and can label quantities appropriately.
- can express numerical answers with a degree of precision appropriate for the problem context.
- can calculate efficiently and accurately.

MP6 - Questions & Prompts

- What mathematical terms apply in this situation?
- How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language, definitions, or properties can you use to explain ...?
- How could you test your solution to see if it answers the problem?

7. Look for and make use of structure.

Students:

- can apply general mathematical rules to specific situations.
- look for the overall structure and patterns in mathematics.
- can simplify complicated things as single objects or as being composed of several objects.

MP7 - Questions & Prompts

- What observations do you make about ...?
- What do you notice about ..., or when ... happens?
- What parts of the problem might you eliminate, or simplify?
- What patterns do you find in ...?
- How do you know if something is a pattern?
- What ideas that we have learned before were useful in solving this problem?
- What are some other problems that are similar to this one?
- How does this relate to ...?
- In what ways does this problem connect to other mathematical concepts?

8. Look for and express regularity in repeated reasoning.

Students:

- recognize repeated calculations, and look for generalizations and shortcuts.
- can see the overall process of the problem, and still attend to the details.
- understand the broader application of patterns, and see the structure in similar situations.
- continually evaluate the reasonableness of their intermediate results.

MP8 - Questions & Prompts

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true?
- How would we prove that ...?
- What do you notice about ...?
- What is happening in this situation?
- What would happen if ...?
- Is there a mathematical rule for ...?
- What predictions or generalizations can this pattern support?
- What mathematical consistencies do you notice?